Indian Statistical Institute
First Semester Examination 2004-2005
B.Math (Hons.) I Year
Probability Theory I

Time: 3hrs   Date:22-11-04   Max Marks : 100

1. If X is uniformly distributed over (-1,1), find
   (a) $P(|X| \geq 1/2)$ [5]
   (b) Let $Y = |X|$. Find the density of Y. [7]

2. The time (in hours) required to repair a machine is an exponentially distributed r.v. with parameter $\lambda = \frac{1}{2}$. What is the conditional probability that a repair takes atleast 10 hours, given that its duration exceeds 9 hours? [7]

3. Let, for every $n \geq 1, S_n$ have the binomial distribution with parameters n and p, $0 < p < 1$. Using the inequality

$$P(S_n \geq k) \leq \frac{kq}{(k-np)^2} \ , \quad k > np$$

deduce the weak law of large numbers: For every $\epsilon > 0$,

$$\lim_{n\to\infty} P\left\{|\frac{S_n}{n} - p| > \epsilon\right\} = 0$$

[10]

4. A standard Cauchy r.v. has density function

$$f(x) = \frac{1}{\pi(1+x^2)} \ , \quad -\infty < x < \infty$$

   (a) Calculate $P(X \leq 1/2)$. [3]
   (b) Show that $\frac{1}{X}$ is also a Cauchy r.v.
   HINT: Calculate the distribution function of $\frac{1}{X}$ [12]

5. Let X and Y be independent and uniformly distributed over the interval (0,1). Compute $P(X \geq Y)$. [10]

6. Let X and Y be independent binomial r.v's with parameters (n_1, p) and (n_2, p) respectively.

   (a) Find the distribution of $X + Y$. [5]

   (b) When $n_1 = n_2 = n$. Calculate the conditional probability that $X = k$ given $X + Y = m$. [5]

7. (a) For an integer valued r.v. N show that

$$EN = \sum_{k=1}^{\infty} P(N \geq k)$$

[5]

(b) Let X be a Poisson r.v with parameter λ. Let $k \geq 0$. Find the value of λ that maximises $P\{X = k\}$ [10]

8. (a) State the De'Moiure - Laplace limit theorem. [5]

(b) A sample is taken in order to find the fraction p of females in a population. What should be the sample size such that the probability of a sampling error less than .005 will .99 or greater. [10]

9. For a simple random walk $s_0 = 0,\ s_1, s_2, \ldots s_{2n}, s_i - s_{i-1} = \pm 1$, show that

$$P\{s_{2n} = 0\} \ = \ P\{s_1 \neq 0, \ldots s_{2n} \neq 0\}$$

[15]

10. Let A_1, A_2, A_3 be 3 events in a probability space. Show that $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$. [5]